METRICAL THEOREMS ON FRACTIONAL PARTS OF SEQUENCES

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1. **Introduction.** Let C be the additive group of real numbers modulo 1, and let $x \to \{x\}$ be the natural mapping from the reals onto C. It is clear what we shall mean by an interval I in C and by the length $l(I)$ of I. Denote the distance of the real number α to the closest integer by $\|\alpha\|$. The image in C of the set of reals ξ satisfying $\|\xi - \theta\| \leqq \varepsilon$ with given θ and $0 < \varepsilon < 1/2$ is an example of an interval of C of length 2ε.

THEOREM 1. *Let $n \geqq 1$ and let $P_1(q), \cdots, P_n(q)$ be nonconstant polynomials with integral coefficients. For each of the integers $j = 1, \cdots, n$ let $I_{j1} \supseteq I_{j2} \supseteq \cdots$ be a sequence of nested intervals in C. Put $\psi(q) = l(I_{1q}) \cdots l(I_{nq})$ and*

$$\Psi(h) = \sum_{q=1}^{h} \psi(q). \tag{1.1}$$

Put $N(h;\alpha_1,\cdots,\alpha_n)$ for the number of integers q, $1 \leqq q \leqq h$, with

$$\{\alpha_j P_j(q)\} \in I_{jq} \qquad (j = 1, \cdots, n). \tag{1.2}$$

Let $\varepsilon > 0$. Then

$$N(h;\alpha_1,\cdots,\alpha_n) = \Psi(h) + O(\Psi(h)^{1/2+\varepsilon}) \tag{1.3}$$

for almost every n-tuple of real numbers $\alpha_1, \cdots, \alpha_n$.

The theorem implies, for example, that the number of solutions of

$$|\alpha q - p - \theta| \leqq q^{-1}$$

in integers p and q, $1 \leqq q \leqq h$, is asymptotically equal to $2 \log h$ for every $\alpha \notin \sigma(\theta)$ where $\sigma(\theta)$ is a set of measure zero. To see this we only have to put $n = 1$, $P(q) = q$ and to define intervals I_q as the images of the sets $\|\xi - \theta\| \leqq q^{-1}$.

On the other hand, let $P(q) = a_0 q^d + \cdots + a_d$ be a polynomial of degree $d > 0$ with integral coefficients, let μ be real, and let $M(h;\alpha)$ be the number of solutions in integers p, q, $1 \leqq q \leqq h$, of

$$|\alpha - p/P(q)| \leqq q^{-\mu}. \tag{1.4}$$

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Then $M(h;\alpha)$ is bounded for almost every α if $\mu > d+1$; $M(h;\alpha) \sim 2|a_0|\log h$ if $\mu = d+1$; and $M(h;\alpha) \sim 2\,|a_0|\,h^{d+1-\mu}(d+1-\mu)^{-1}$ for almost every α if $\mu < d+1$.

To see this, we remark that for $\mu > d$ and large q, (1.4) is equivalent to $\|\alpha P(q)\| \leqq |P(q)|\,q^{-\mu}$. Thus our interval I_q has length $\psi(q) = 2|P(q)|\,q^{-\mu} = |2a_0q^{d-\mu} + 2a_1q^{d-\mu-1} + \cdots|$, and the theorem gives the result. For $\mu = d$, (1.4) becomes $|\alpha P(q) - p| \leqq |a_0 + a_1q^{-1} + \cdots|$, and $M(h;\alpha)$ becomes $2|a_0|h$ plus (or minus) the number of solutions of $\|\alpha P(q)\| \leqq |a_1q^{-1} + \cdots|$ for $1 \leqq q \leqq h$, whence $M(h;\alpha) \sim 2|a_0|h$ almost everywhere. Finally for $\mu < d$ our formula for $M(h;\alpha)$ is in fact true for every α. The reader should have no difficulty in proving this elementary result.

There can be at most countably many α_j's such that $\{\alpha_jP_j(q)\}$ is an endpoint of I_{jq} for some q, and hence we may assume I_{jq} to be closed ($j = 1,\cdots,n$; $q = 1,2,\cdots$). The intersections $J_j = \bigcap_q I_{jq}$ ($j = 1,\cdots,n$) are then nonempty. The case where $0 \in J_j$ for each j is usually called the homogeneous case, the general case the inhomogeneous case.

Our theorem implies in particular that $N(h;\alpha_1,\cdots,\alpha_n)$ remains bounded almost everywhere if $\Psi(h)$ is bounded, while it will tend to infinity almost everywhere if $\Psi(h)$ tends to infinity. This had been proved by Khintchine [9] in the homogeneous case under the assumption that $P_j(q) = q$ ($j = 1,\cdots,n$) and that $q\psi(q)$ is decreasing. Szüsz [13] generalized Khintchine's result to the inhomogeneous case. Szüsz' method involves continued fractions and therefore applies only to the case $n = 1$. Before Szüsz, Cassels [2] had shown that Khintchine's conclusion is true for "almost every inhomogeneous case," that is, if $(I_{1q},\cdots,I_{nq})$ is replaced by its translation by a vector $(\theta_1,\cdots,\theta_n)$ of reals mod 1 ($q = 1,2,\cdots$), then the conclusion is true for almost every $\theta_1,\cdots,\theta_n$. Thus Cassel's result was "doubly metrical."

Erdös [5] proved for the homogeneous case with $n = 1$, $P(q) = q$, that $N(h;\alpha) \sim \Psi(h)$ almost everywhere, and the author [12](2) proved (1.3) in this case. Our generalization from the homogeneous to the inhomogeneous case is not trivial. We shall choose $\theta_j \in J_j$ ($j = 1,\cdots,n$) and use rational approximations to θ_j. The generalization from linear to general polynomials also causes some difficulty.

Le Veque [10] proved a general theorem where polynomials $P(q)$ are replaced by general sequences $a(q)$ which have to satisfy a certain condition. However, this condition is not satisfied for $a(q) = q$, and it is difficult to decide whether it is satisfied for nonlinear polynomials.

It would be possible to replace (1.2) by $(\{\alpha_1P_1(q)\},\cdots,\{\alpha_nP_n(q)\}) \in H_q$, thus replacing products of intervals $I_{1q} \times \cdots \times I_{nq}$ by somewhat more general sets H_q of $C \times \cdots \times C$.

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(2) We use this opportunity to mention two errors in [12]: In Theorem 1 of [12] one has to assume that the functions $\psi_j(q)$ are bounded. Everywhere in §6 except in $\beta(Q,\theta)$, θ should be replaced by $\Theta = (\theta_1,\ldots,\theta_u)$.

In §10 we shall point out how one could prove a more general theorem where the expressions $\alpha_j P_j(q)$ are replaced by linear forms $\alpha_{j1}P_{j1}(q_1)+\cdots+\alpha_{jm}P_{jm}(q_m)$. A special case of such a result is contained in Theorem 2 of [12].

THEOREM 2. *Let a sequence of positive integers $a_i(1) < a_i(2) < \cdots$ be given for $i = 1,\cdots,n$. Let θ be arbitrary but fixed, and put*

$$\sum (h;\alpha_1,\cdots,\alpha_n) = \sum_{q_1=1}^{h} \cdots \sum_{q_n=1}^{h} \left(q_1 \cdots q_n \left\| \sum_{i=1}^{n} \alpha_i a_i(q_i) + \theta \right\| \right)^{-1} .$$

Then one has for $\varepsilon > 0$ and almost every $\alpha_1,\cdots,\alpha_n$

$$(\log h)^{n+1} \ll \sum(h;\alpha_1,\cdots,\alpha_n) \ll (\log h)^{n+1+\varepsilon}. \tag{1.5}$$

Using Theorem 2, together with an n-dimensional generalization of a result of Erdös and Turan [7, Theorem 3], we shall easily deduce

THEOREM 3. *Assume the hypotheses of Theorem 1 to be satisfiied, and assume we deal with the special case $P_j(q) = q$ $(j = 1,\cdots,n)$ and $I_{j1} = I_{j2} = \cdots$ $(j = 1,\cdots,n)$. Write ψ for $\psi(1) = \psi(2) = \cdots$, and let $\varepsilon > 0$. Then*

$$N(h;\alpha_1,\cdots,\alpha_n) = h\psi + O(\log h)^{n+1+\varepsilon}$$

for almost every $\alpha_1,\cdots,\alpha_n$.

Khintchine [8, §3], proved the surprisingly small error-term $O(\log h)^{1+\varepsilon}$ for $n = 1$, and hence our result is not best possible. However, Khintchine's method involves continued fractions and cannot easily be generalized to $n > 1$. It seems that Theorem 1 cannot much be improved for nonlinear polynomials. Behnke [1, Theorem XXV] showed for $n = 1$, $P(q) = q^2$ and $I_1 = I_2 = \cdots = I$, say, that the relation $D_\alpha(h) =^{\text{def}} \sup_I |N_I(h;\alpha) - hl(I)| \ll \sqrt{h}$ is wrong for every α.

2. **Notation and simplification.** Throughout, $[\alpha]$ will the integral part of the real number α. U will denote the unit interval $0 \leqq \xi < 1$.

We shall prove the case $n = 1$ of Theorem 1 in §§2–8. In §9 we shall point out the necessary changes for $n > 1$.

The set of α's in U where $\{\alpha P(q)\} \in I_q$ has measure $\psi(q)$. Assume now that $\Psi(h)$ is bounded. Given $\varepsilon > 0$ there is a q_0 such that $\sum_{q>q_0}\psi(q) < \varepsilon$, and the set of α's in U such that $\{\alpha P(q)\} \in I_q$ for some $q > q_0$ has measure $< \varepsilon$. Hence $N(h;\alpha)$ is bounded for almost every α.

From now on, we shall assume that $\Psi(h)$ tends to infinity.

Let $\theta \in J = \bigcap I_q$. Then each I_q is union of θ and of two intervals I_q^l and I_q^r, where I_q^l is of the type $0 < \{\theta - \xi\} \leqq \psi^l(q)$, where I_q^r is of the type $0 < \{\xi - \theta\} \leqq \psi^r(q)$, and where $\psi^l(q) + \psi^r(q) = \psi(q)$. ($I_q^l$ or I_q^r may be empty.) Now $\Psi^l(h)$, $\Psi^r(h)$, $N^l(h;\alpha)$, $N^r(h;\alpha)$ can be defined in the obvious way. One has $\Psi(h) = \Psi^l(h) + \Psi^r(h)$ and $N(h;\alpha) = N^l(h;\alpha) + N^r(h;\alpha)$ for almost every α. Hence it will suffice to prove the theorem for the case of intervals of type I^l and the case of intervals of type I^r.

Since the mapping $\xi \to -\xi$, $\theta \to -\theta$ transforms intervals of type I^l into intervals of type I^r, we may restrict ourselves to intervals of type I^r.

From now on, I_q will denote the interval

$$0 < \{\xi - \theta\} \leqq \psi(q).$$

Replacing $P(q)$ by $-P(q)$ and α by $-\alpha$ if necessary, we may assume that $P(q) > 0$, $P'(q) > 0$ for $q > q_0$. Making a translation by q_0 we may even assume $P(q) > 0$, $P'(q) > 0$ for $q > 0$.

The introduction of a parameter k is essential for our proof. Put $\phi(k,x)$ for the number of integers y between 1 and x, $1 \leqq y \leqq x$, such that g.c.d.$(x,y) \leqq k$. $\phi(1,x)$ is the well-known Euler ϕ-function.

Given $q \geqq 1$ there are pairs of relatively prime integers a, b such that

$$1 \leqq a \leqq q^{1/2} \text{ and } |\theta - b/a| < a^{-1} q^{-1/2}. \tag{2.1}$$

This follows from Dirichlet's principle. For every integer $q \geqq 1$ we pick integers $a = a(q)$, $b = b(q)$ with these properties. We define $S(k,q)$ as the set of integers p where

$$\text{g.c.d.}(pa(q) + b(q), P(q)) \leqq k. \tag{2.2}$$

The sets $S(k,q)$ have two important properties:

(1) If $p \in S(k,q)$ and $p \equiv p' (\operatorname{mod} P(q))$, then $p' \in S(k,q)$.

(2) The number $\phi^*(k,q)$ of integers of $S(k,q)$ in $1 \leqq x \leqq P(q)$ satisfies $\phi^*(k,q) \geqq \phi(k,P(q))$.

To prove (2), put $P(q) = q_1 q_2$ where every prime factor of q_1 divides a and where q_2 and a are relatively prime. Now g.c.d.$(a, b) = 1$ yields g.c.d.$(pa + b, P(q)) =$ g.c.d.$(pa + b, q_2)$ and $\phi^*(k,q) = q_1 \phi(k, q_2) \geqq \phi(k, P(q))$.

We now put

$$\beta(q,\alpha) = \begin{cases} 1 \text{ if } \alpha \in U \text{ and } \{\alpha\} \in I_q, \\ 0 \text{ otherwise,} \end{cases}$$

$$\gamma(q,\alpha) = \sum_p \beta(q, \alpha P(q) - p),$$

$$\gamma(k,q,\alpha) = \sum_{p \in S(k,q)} \beta(q, \alpha P(q) - p),$$

$$\Gamma(q) = \int_0^1 \gamma(q,\alpha) d\alpha,$$

$$\Gamma(k,q) = \int_0^1 \gamma(k,q,\alpha) d\alpha,$$

$$\Gamma(k,q,r) = \int_0^1 \gamma(k,q,\alpha)\gamma(k,r,\alpha) d\alpha,$$

$$A(k,q,r) = \Gamma(k,q,r) - \psi(q)\psi(r),$$

and

$$\Psi(u,v) = \sum_{q=u+1}^{v} \psi(q).$$

It is easy to see that $N(h;\alpha) = \sum_{q=1}^{h} \gamma(q,\alpha)$, and we define

$$N(k;u,v;\alpha) = \sum_{q=u+1}^{v} \gamma(k,q,\alpha).$$

One has

$$\Gamma(q) = \sum_{p} \int_{0}^{1} \beta(q,P(q)\alpha - p)d\alpha = P(q)\int_{-\infty}^{\infty} \beta(q,P(q)\alpha)d\alpha = \psi(q), \tag{2.3}$$

and similarly

$$\Gamma(k,q) = \psi(q)\phi^*(k,q)P(q)^{-1}. \tag{2.4}$$

Summing over q we find

$$\int_{0}^{1} N(h;\alpha)d\alpha = \Psi(h) \tag{2.5}$$

and

$$\int_{0}^{1} N(k;u,v;\alpha)d\alpha = \sum_{q=u+1}^{v} \psi(q)\phi^*(k,q)P(q)^{-1}. \tag{2.6}$$

3. Deduction of Theorem 1 from two propositions.

PROPOSITION 1. *Let $\delta > 0$. Then*

$$\sum_{q=1}^{h} (P(q) - \phi(k,P(q)))P(q)^{-1} \ll hk^{\delta-1} + h^{\delta}k^{\delta}. \tag{3.1}$$

PROPOSITION 2. *For every $\delta > 0$*

$$\sum_{q=1}^{h} \sum_{r=1}^{h} A(k,q,r) \ll \Psi^{1+\delta}(h) + \Psi(h)k^{\delta}. \tag{3.2}$$

REMARK. Here and later, the estimate $\ll$ holds simultaneously in h and k. That is, the constant implied by $\ll$ depends only on δ.

We are going to show that Theorem 1 is a consequence of these two propositions. The propositions will be proved later.

LEMMA 1. *Let $\omega_1(q)$, $\omega_2(q)$, $\omega_3(q)$ be positive bounded functions of positive integers q, and put*

$$\Omega_i(h) = \sum_{q=1}^{h} \omega_i(q) \qquad (i = 1,2,3).$$

Assume that ω_1 and ω_3 are decreasing, and that $\Omega_2(r) \leqq \Omega_3(r)$ for every r. Then

$$\sum_{q=1}^{h} \omega_1(q)\omega_2(q) \ll \Omega_3([\Omega_1(h)]). \tag{3.3}$$

Proof. If Ω_1 is bounded, then so is the sum in (3.3). Hence we assume Ω_1 to be unbounded. Since ω_1 is decreasing, and since $\Omega_2 \leqq \Omega_3$, one finds by partial summation that $\sum_{q=1}^{h} \omega_1(q)\omega_2(q) \leqq \sum_{q=1}^{h} \omega_1(q)\omega_3(q)$.

To estimate the latter sum we may assume $\omega_1(q) \leqq 1$. Put $m_0 = 0$ and for integral $a > 0$ put m_a for the largest m with $\Omega_1(m) \leqq a$. Then $m_a \geqq a$ and $\omega_1(m_a + 1) + \cdots + \omega_1(m_{a+1}) \leqq 2$. Putting $b = [\Omega_1(h)]$ we obtain

$$\sum_{q=1}^{h} \omega_1(q)\omega_3(q) \leqq \sum_{a=0}^{b} (\omega_1(m_a + 1)\omega_3(m_a + 1) + \cdots + \omega_1(m_{a+1})\omega_3(m_{a+1}))$$

$$\leqq 2\sum_{a=0}^{b} \omega_3(m_a + 1) \leqq 2\sum_{a=1}^{b+1} \omega_3(a) = 2\Omega_3([\Omega_1(h)] + 1).$$

Denote by J_r the set of intervals $(u, v]$, $0 \leqq u = t \cdot 2^s < v = (t+1)2^s \leqq 2^r$ where r, s, t are non-negative integers. Every interval $(0, w]$ where w is integral and $w \leqq 2^r$ is union of not more than $\max(1, r)$ intervals of J_r. Given an integer $u > 0$ put n_u for some integer satisfying $[\Psi(n_u)] = u$, and put $n_0 = 0$. Since $\psi(q) \leqq 1$ and since $\Psi(h)$ tends to infinity, such an n_u will always exist. Put $h_r = n_{2^r}$.

For the remainder of this section, k and r will be connected by

$$k = 2^r. \tag{3.4}$$

LEMMA 2. *Let $\delta > 0$. Then*

$$0 \leqq \int_0^1 (N(h_r;\alpha) - N(k;0,h_r;\alpha))d\alpha \ll 2^{r\delta} \tag{3.5}$$

and

$$\sum_{(u,v]\in J_r} \int_0^1 (N(k;n_u,n_v;\alpha) - \Psi(n_u,n_v))^2 d\alpha \ll 2^{r+r\delta}. \tag{3.6}$$

Proof. Formulae (2.5) and (2.6) yield

$$S_r \equiv \int_0^1 (N(h_r;\alpha) - N(k;0,h_r;\alpha))d\alpha = \sum_{q=1}^{h_r} \psi(q)(P(q) - \phi^*(k,q))P(q)^{-1}$$

$$\leqq \sum_{q=1}^{h_r} \psi(q)(P(q) - \phi(k,P(q)))P(q)^{-1}.$$

We now put

$$\omega_1(q) = \psi(q),\ \omega_2(q) = (P(q) - \phi(k,P(q)))P(q)^{-1},\ \omega_3(q) = c(k^{\delta-1} + q^{\delta-1}k^{\delta}).$$

Proposition 1 shows that Lemma 1 is applicable if $c > 0$ is chosen large enough. Under our conditions we actually obtain the bound $2\Omega_3([\Omega_1(h))] + 1)$. Hence

$$S_r \ll \Psi(h_r)k^{\delta-1} + \Psi(h_r)^\delta k^\delta \ll 2^{r+r(\delta-1)} + 2^{2r\delta} \ll 2^{2r\delta}.$$

This is true for every $\delta > 0$, and hence (3.5) is proved.

$$N(k;u,v;\alpha) - \Psi(u,v) = \sum_{q=u+1}^{v} (\gamma(k,q,\alpha) - \psi(q)).$$

Hence by (2.3), (2.4) and the estimate just derived,

$$\int_0^1 (N(k;u,v;\alpha) - \Psi(u,v))^2 d\alpha$$

$$= \sum_{q=u+1}^{v} \sum_{q'=u+1}^{v} (\Gamma(k,q,q') - \Gamma(k,q)\psi(q') - \Gamma(k,q')\psi(q) + \psi(q)\psi(q'))$$

$$= \sum_{q=u+1}^{v} \sum_{q'=u+1}^{v} A(k,q,q') + 2 \sum_{q=u+1}^{v} \sum_{q'=u+1}^{v} \psi(q)\psi(q')(P(q) - \phi^*(k,q))P(q)^{-1}$$

$$\ll \sum_{q=u+1}^{v} \sum_{q'=u+1}^{v} A(k,q,q') + \sum_{q'=u+1}^{v} \psi(q')2^{r\delta}.$$

We first consider the part of the sum (3.6) where $(u,v]$ are intervals of J_r with fixed s (see the definition of J_r). These intervals cover $(0,2^r]$ exactly once, and hence the corresponding intervals $(n_u,n_v]$ cover $(0,h_r]$ exactly once. Our part of the sum (3.6) has the upper bound

$$\sum_{q=1}^{h_r} \sum_{q'=1}^{h_r} A(k,q,q') + 2^{r\delta} \sum_{q=1}^{h_r} \psi(q) \ll \Psi(h_r)^{1+\delta} + \Psi(h_r)k^\delta + 2^{r\delta}\Psi(h_r) \ll 2^{r+r\delta}.$$

Summing over s from 0 to r we find the bound $\ll r2^{r+r\delta}$ for the sum (3.6). Since $\delta > 0$ is arbitrary, Lemma 2 is proved.

Lemma 3. *Let δ be positive and fixed. Then there is a sequence of subsets $\sigma_1, \sigma_2, \cdots$ of U with measures*

$$\mu_r = \int_{\sigma r} d\alpha \ll r^{-2}$$

such that

$$N(n_w;\alpha) = \Psi(n_w) + O(r^2 2^{r/2+r\delta})$$

for every $w \leqq 2^r$ and every $\alpha \in U$ which is not in σ_r.

Proof. We define σ_r to be the subset of U where not both of the following inequalities hold:

$$0 \leqq N(h_r;\alpha) - N(k;0,h_r;\alpha) \leqq r^2 2^{r/2}, \tag{3.7}$$

$$\sum_{(u,v]\in J_r} (N(k;n_u,n_v;\alpha) - \Psi(n_u,n_v))^2 \leqq r^2 2^{r+r\delta}. \tag{3.8}$$

Lemma 2 implies $\mu_r \ll r^{-2}$. Every interval $(0,w]$, $w \leqq 2^r$, is union of at most $\max(1,r)$ intervals of J_r, hence $(0,n_w]$ is union at most $\max(1,r)$ intervals $(n_u,n_v]$ where $(u,v] \in J_r$. Thus $N(k;0,n_w;\alpha) - \Psi(n_w) = \sum (N(k;n_u,n_v;\alpha) - \Psi(n_u,n_v))$, where the sum is over at most $r+1$ pairs $(u,v] \in J_r$. This relation together with (3.8) and Cauchy's inequality gives for $\alpha \in U$, $\alpha \notin \sigma_r$

$$(N(k;0,n_w;\alpha) - \Psi(n_w))^2 \leqq r^2(r+1)2^{r+r\delta}. \tag{3.9}$$

Lemma 3 is a consequence of (3.7) and (3.9).

Proof of Theorem 1. Since $\sum r^{-2}$ is convergent, there exists for almost every $\alpha \in U$ an $r_0 = r_0(\alpha)$ such that $\alpha \notin \sigma_r$ for $r \geqq r_0$. Assume α has such an r_0, and assume $w > 2^{r_0}$. Choose r such that $2^{r-1} \leqq w < 2^r$. Then $r > r_0$, $\alpha \notin \sigma_r$, and Lemma 3 implies

$$\begin{aligned} N(n_w;\alpha) &= \Psi(n_w) + O(r^2 2^{r/2+r\delta}) \\ &= \Psi(n_w) + O(w^{1/2+\delta}\log^2 w) \\ &= \Psi(n_w) + O(\Psi^{1/2+\delta}(n_w)\log^2\Psi(n_w)). \end{aligned} \tag{3.10}$$

Since $\Psi(n_{w+1}) = \Psi(n_w) + O(1)$, (3.10) is true for arbitrary integers h and not only the n_w's. And since $\delta > 0$ was arbitrary, we find

$$N(h;\alpha) = \Psi(h) + O(\Psi(h)^{1/2+\varepsilon})$$

for almost every $\alpha \in U$. Hence (1.3) is true for almost every α.

4. **The number of solutions of $P(x) \equiv 0 \pmod{d}$.** Put $D(q)$ for the number of positive divisors of q. As is well known,

$$D(q) \ll q^\delta \tag{4.1}$$

for every $\delta > 0$. Put $z(d) = z_P(d)$ for the number of solutions of $P(x) \equiv 0 \pmod{d}$. Here, as always, $P(x)$ is a nonconstant polynomial with integral coefficients. Define the discriminant Δ of $P(x)$ in the usual way if $P(x)$ is nonlinear, and put $\Delta = \alpha_0$ if $P(x) = a_0 x + a_1$.

LEMMA 4. *Let $P(x)$ be a polynomial of degree f and with discriminant $\Delta \neq 0$. Then $z_P(p^k) \leqq f\Delta^2$ for every prime-power p^k.*

Proof. For linear $P(x)$ it is well known that $z(m) \leqq \text{g.c.d.}(m,\Delta) \leqq \Delta \leqq f\Delta^2$. The case where $P(x)$ is nonlinear and primitive, that is, where the coefficients of $P(x)$ are relatively prime, is Theorem 54 of [11]. A proof can be found there. In the general nonlinear case one has $P(x) = cQ(x)$ with primitive $Q(x)$, whence $z_P(p^k) \leqq cz_Q(p^k) \leqq cf\Delta^2{}_Q \leqq f\Delta^2{}_P$.

COROLLARY. *Let $P(x)$ be a polynomial with no multiple factors. Let $\delta > 0$. Then*

(4.2) $$z_P(d) \ll d^{\delta}.$$

Proof. The set τ of prime-powers p^k such that $p^{k\delta} \geqq f\Delta^2$ is fiinete. For every d,

$$z(d)d^{-\delta} \leqq \prod_{p^k \in \tau} z(p^k)p^{-k\delta} \ll 1.$$

Given an integer $g > 0$ we define a function ${}^g(d)$ of positive integers d as follows: ${}^g(d)$ is multiplicative, and ${}^g(p^{gx+y}) = p^{x+1}$ if p is a prime and $1 \leqq y \leqq g$. Our function has the property that $d \mid m^g$ implies ${}^g(d) \mid m$.

LEMMA 5. *Let $P(x)$ be a nonconstant polynomial, g a positive integer and $s > 1$. Then the two sums*

(4.3) $$\sum_{d=1}^{\infty} z_P(d)d^{-s}$$

and

(4.4) $$\sum_{d=1}^{\infty} ({}^g(d))^{-s}$$

are convergent.

Proof. There is an integer m and a polynomial $Q(x)$ without multiple factors such that $P(x) \mid Q(x)^m$. Now $P(x) \equiv 0 \pmod d$ implies $Q(x) \equiv 0({}^m(d))$, and hence $z_P(d)d^{-1} \leqq z_Q({}^m(d))({}^m(d))^{-1}$. Thus one has for $d = p^{mx+y}$ where p is prime and $0 < y \leqq m$,

$$\begin{aligned} z_P(d)d^{-s} &\leqq z_Q({}^m(d))({}^m(d))^{-1}d^{1-s} = z_Q(p^{x+1})p^{(1-s)(mx+y)-x-1} \\ &\leqq f\Delta_Q^2 p^{-x(m(s-1)+1)-y(s-1)-1} \leqq f\Delta_Q^2 p^{-sx-s}. \end{aligned}$$

This implies

$$\sum_{e=1}^{\infty} z_P(p^e)p^{-es} \leqq mf\Delta_Q^2 p_J^{-s} \sum_{x=0}^{\infty} p^{-xs} \leqq c_s p^{-s}.$$

Since the product $\prod_p(1 + cp^{-s})$ over all primes p is convergent, the convergence of (4.3) follows.

The convergence of (4.4) is proved similarly.

5. **Proof of Proposition 1.** The Euler ϕ-function $\phi(x) = \phi(1,x)$ can be expressed $\phi(x) = x\sum_{y \mid x}\mu(y)y^{-1}$, where $\mu(y)$ is the Moebius function. Now

$$\phi(k, P(q)) = \sum_{x \leqq k;\, x \mid P(q)} \phi(P(q)x^{-1}) = \sum_{x \leqq k;\, x \mid P(q)} P(q)x^{-1} \sum_{y \mid P(q)x^{-1}} \mu(y)y^{-1};$$

hence

$$\begin{aligned} T_{k,h} &= \sum_{q=1}^{h} \phi(k, P(q))P(q)^{-1} = \sum_{q=1}^{h} \sum_{x \leqq k;\, x \mid P(q)} x^{-1} \sum_{y \mid P(q)x^{-1}} \mu(y)y^{-1}, \\ &= \sum_{x \leqq k;\, x \leqq P(h)} x^{-1} \sum_{y \leqq P(h)x^{-1}} \mu(y)y^{-1} \sum_{q \leqq h\cdot\, xy \mid P(q)} 1. \end{aligned}$$

The number of $q \leqq h$ such that $xy \mid P(q)$ equals $hz(xy)(xy)^{-1} + O(z(xy))$. Therefore

$$T_{k,h} = h \sum_{x \leqq k;\, x \leqq P(h)} \sum_{y \leqq P(h)x^{-1}} z(xy)(xy)^{-2}\mu(y) + O\Bigl(\sum_{x \leqq k} \sum_{y \leqq P(h)} z(xy)(xy)^{-1}\Bigr)$$
$$= hU_{k,h} + O(V_{k,h}),$$

say. Putting $xy = w$ and using (4.1) with $\delta = \varepsilon/2$ and Lemma 5 with $s = 1 + \varepsilon/2$, $\varepsilon > 0$, we find

$$U_{k,h} = \sum_{w \leqq k;\, w \leqq P(h)} z(w)w^{-2} \sum_{y \mid w} \mu(y) + O\Bigl(\sum_{w > k} z(w)w^{-2}D(w)\Bigr)$$
$$= 1 + O\Bigl(k^{\varepsilon-1}\Bigl(\sum_{w=1}^{\infty} z(w)w^{-1-\varepsilon/2}D(w)w^{-\varepsilon/2}\Bigr)\Bigr) = 1 + O(k^{\varepsilon-1}).$$

Similarly,

$$V_{k,h} \leqq \sum_{w \leqq P(h)k} z(w)w^{-1}D(w) \leqq P(h)^{\varepsilon}k^{\varepsilon} \sum_{w=1}^{\infty} (z(w)w^{-1-\varepsilon/2}D(w)w^{-\varepsilon/2}) \ll P(h)^{\varepsilon}k^{\varepsilon}.$$

Combining our formulae and observing that $\varepsilon > 0$ was arbitrary we obtain $T_{k,h} = h + O(hk^{\delta-1} + h^{\delta}k^{\delta})$, thereby proving the proposition.

We use the remainder of this section to prove four related lemmas.

LEMMA 6. *Let $P(x)$ be a polynomial of degree $f > 1$, and let $\varepsilon > 0$. Then*

$$W_h \equiv \sum_{q=1}^{h} P(q)^{-1} \sum_{d \mid P(q);\, d < q^{f-\varepsilon}} d \sum_{r \leqq q;\, d \mid P(r)} 1 \ll h.$$

Proof. Choose $\delta > 0$ so small that $2\delta f \leqq \varepsilon(1 - f^{-1})$.

There is an integer $g \geqq 1$ and a polynomial $Q(x)$ with no multiple factors such that $P(x) \mid Q(x)^g$. We may choose $g \leqq f$. Now $d \mid P(r)$ implies ${}^g(d) \mid Q(r)$, hence the number of $r \leqq q$ with $d \mid P(r)$ is not larger than $(q({}^g(d))^{-1} + 1)z_Q({}^g(d))$ and therefore by the corollary to Lemma 4 not larger than

$$\ll (q({}^g(d))^{-1} + 1)\, d^{\delta} \leqq (qd^{-1/g} + 1)\, d^{\delta} \leqq (qd^{-1/f} + 1)\, d^{\delta}.$$

Using $D(P(q)) \ll q^{f\delta}$ we obtain

$$W_h \ll \sum_{q=1}^{h} q^{-f} \sum_{d \mid P(q);\, d < q^{f-\varepsilon}} (qd^{1-1/f+\delta} + d^{1+\delta})$$
$$\ll \sum_{q=1}^{h} D(P(q))(q^{-f+1+(1-1/f+\delta)(f-\varepsilon)} + q^{-f+(1+\delta)(f-\varepsilon)})$$
$$\ll \sum_{q=1}^{h} q^{2\delta f - \varepsilon(1-1/f)} \ll h.$$

LEMMA 7. *Let $P(x)$ be arbitrary and $\delta > 0$. Then $\sum_{q=1}^{h} \sum_{d \mid P(q)} d^{-\delta} \ll h$.*

Proof. The part of the sum where $d \geqq q$ is not larger than

$$\sum_{q=1}^{h} q^{-\delta} D(P(q)) \ll h.$$

The part of the sum where $d < q$ is estimated by

$$\sum_{d=1}^{h} d^{-\delta} \sum_{d<q\leqq h;\, d|P(q)} 1 \leqq \sum_{d=1}^{h} d^{-\delta} h d^{-1} z(d) \leqq h \sum_{d=1}^{\infty} z(d) d^{-1-\delta} \ll h.$$

LEMMA 8. *Write $D_k(x)$ for the number of positive divisors of x which are not larger than k, and let $\delta > 0$. Then*

$$\sum_{q=1}^{h} D_k(P(q)) \ll hk^{\delta}.$$

Proof. We break the sum into two parts, $\sum_{q=1}^{\min(k,h)} + \sum_{k<q\leqq h}$, where the second part may be empty. For q contributing to the first part of the sum, $D_k(P(q)) \leqq D(P(q)) \ll k^{\delta}$, and we obtain the desired estimate. The second part equals

$$\sum_{k<q\leqq h} D_k(P(q)) = \sum_{d\leqq k} \sum_{k<q\leqq h;\, d|P(q)} 1 \leqq \sum_{d\leqq k} hd^{-1} z(d)$$

$$\leqq hk^{\delta} \sum_{d=1}^{\infty} z(d) d^{-1-\delta} \ll hk^{\delta}.$$

LEMMA 9. *Write $D(x,y)$ for the number of common positive divisors of integers $x, y \neq 0, 0$. Let $P_1(x), P_2(x)$ be polynomials with integral coefficients such that $P_i(x) \neq 0$ for $x > 0$. Then*

$$X_{h_1,h_2} \equiv \sum_{q_1=1}^{h_1} \sum_{q_2=1}^{h_2} D(P_1(q_1), P_2(q_2)) \ll h_1 h_2. \tag{5.1}$$

This estimate holds simultaneously in h_1, h_2.

Proof. It is sufficient to prove (5.1) with $P_1(x)$, $P_2(x)$ both replaced by the product $P_1(x)P_2(x)$. We may therefore assume $P_1(x) = P_2(x) = P(x)$, say. There is an integer $g > 0$ and a polynomial $Q(x)$ without multiple factors such that $P(x) \mid Q(x)^g$.

Let σ be the set of positive divisors of $P(x)$ where $1 \leqq x \leqq \min(h_1, h_2)$. The number of elements of σ is $\ll (\min(h_1, h_2))^{1+\delta}$ for every $\delta > 0$.

$$X_{h_1,h_2} \leqq \sum_{d\in\sigma} \left(\sum_{q_1\leqq h_1;\, d|P(q_1)} 1 \right) \left(\sum_{q_2\leqq h_2;\, d|P(q_2)} 1 \right)$$

$$\leqq \sum_{d\in\sigma} \left(\sum_{q_1\leqq h_1;\, {}^g(d)|Q(q_1)} 1 \right) \left(\sum_{q_2\leqq h_2;\, {}^g(d)|Q(q)_2} 1 \right)$$

$$\leqq \sum_{d\in\sigma} (h_1({}^g(d))^{-1} + 1)(h_2({}^g(d))^{-1} + 1) z_Q^2({}^g(d))$$

$$\ll \sum_{d\in\sigma} (h_1({}^g(d))^{-1} + 1)(h_2({}^g(d))^{-1} + 1)({}^g(d))^{2\delta}.$$

Using the distributive law we can break this sum into four parts, and Lemma 5 implies that each part is $\ll h_1 h_2$.

6. **Estimates for** $A(k, q, r)$. In what follows, $d^* = d^*(q, r)$ will mean g.c.d.$(P(q), P(r))$. Put $B(k,q,r)$ for the number of pairs of integers p, s, $p \in S(k,q)$, $s \in S(k,r)$, $0 \leqq p < P(q)$, such that

$$|P(q)(s+\theta) - P(r)(p+\theta)| < \min(d^*, P(q)\psi(r)).$$

LEMMA 10. *For* $r \leqq q$, $A(k,q,r) \leqq \psi(q)P(q)^{-1}B(k,q,r)$.

Proof. All the expressions $P(q)s - P(r)p$ are multiples of d^*. Write $C(l,k,q,r)$ for the number of pairs p, s, $p \in S(k,q)$, $s \in S(k,r)$, $0 \leqq p < P(q)$ such that $P(q)s - P(r)p = ld^*$. The congruence $P(r)p \equiv ld^* \pmod{P(q)}$ has d^* solutions in p, and therefore

$$C(l,k,q,r) \leqq d^*.$$

By definition,

$$\Gamma(k,q,r) = \sum_{p \in S(k,q)} \sum_{s \in S(k,r)} \int_0^1 \beta(q, P(q)\alpha - p)\beta(r, P(r)\alpha - s)\,d\alpha.$$

We now make the substitution $P(q)\alpha' = P(q)\alpha - p - \theta$. Then $P(r)\alpha - s = P(r)\alpha' + \theta - (P(q)(s+\theta) - P(r)(p+\theta))P(q)^{-1}$ and

$$\begin{aligned}
\Gamma(k,q,r) &= \sum_{p \in S(k,q)} \sum_{s \in S(k,r)} \int_{-(p+\theta)P(q)^{-1}}^{1-(p+\theta)P(q)^{-1}} \beta(q, P(q)\alpha' + \theta)\beta(r, P(r)\alpha' + \theta \\
&\qquad\qquad - (P(q)(s+\theta) - P(r)(p+\theta))P(q)^{-1})\,d\alpha' \\
&= \sum_l C(l,k,q,r) \\
&\quad \cdot \int_{-\infty}^{\infty} \beta(q, P(q)\alpha + \theta)\beta(r, P(r)\alpha + \theta - ld^* + (P(q) - P(r))\theta)P(q)^{-1})\,d\alpha \\
&= \sum_l C(l,k,q,r) D(q, r, ld^* + (P(q) - P(r))\theta),
\end{aligned}$$

where $D(q,r,t) = \int_{-\infty}^{\infty} \beta(q, P(q)\alpha + \theta)\beta(r, P(r)\alpha + \theta - tP(q)^{-1})d\alpha$.

For the following estimates we recall that $\beta(q, \xi + \theta)$ is the characteristic function of $0 < \xi \leqq \psi(q)$. We note

$$\int_{-\infty}^{\infty} D(q,r,t)dt = \psi(q)\psi(r)$$

as well as $0 \leqq D(q,r,t) \leqq \psi(q)P(q)^{-1}$ and the fact that D is zero outside the interval $(-P(q)\psi(r),\ P(r)\psi(q))$, hence in particular if $|t| \geqq P(q)\psi(r)$. Furthermore, $D(q,r,t)$ is decreasing for $t > 0$, increasing for $t < 0$. Hence

$$
\begin{aligned}
\Gamma(k,q,r) \leqq\ & d^* \sum_{l;\ |ld^*+(P(q)-P(r))\theta| \geqq d^*} D(q,r,ld^*+(P(q)-P(r))\theta) \\
& + \sum_{l;\,|ld^*+(P(q)-P(r))\theta|<d^*} C(l,k,q,r)\,D(q,r,ld^*+(P(q)-P(r))\theta) \\
& \leqq d^*\int_{-\infty}^{\infty} D(q,r,\lambda d^*+(P(q)-P(r))\theta)d\lambda + \psi(q)P(q)^{-1}B(k,q,r) \\
& = \psi(q)\psi(r) + \psi(q)P(q)^{-1}B(k,q,r),
\end{aligned} \tag{6.1}
$$

and the lemma follows.

Put $E_\delta(k,q,r)$ for the number of $p \in S(k,q)$, $s \in S(k,r)$, $0 \leqq p < P(q)$ with $|P(q)(s+\theta) - P(r)(p+\theta)| < P(q)q^{-1}d^{*\delta}$.

LEMMA 11. *Let $P(q)$ be a polynomial of degree $f > 0$, and let $\varepsilon = 1$ if $f = 1$, $\varepsilon > 0$ if $f > 1$. Let $\delta > 0$. Then*

$$
\begin{aligned}
\sum_{q=1}^{h}\sum_{r=1}^{q} \psi(q)P(q)^{-1}B(k,q,r) \ll\ & \Psi(h)^{1+\delta} \\
& + \sum_{q=1}^{h}\ \sum_{r\leqq q \text{ with } d^*\geqq q^{f-\varepsilon}} \psi(q)P(q)^{-1}E_\delta(k,q,r).
\end{aligned} \tag{6.2}
$$

Proof. Choose $\delta_1 > 0$, $\delta_2 > 0$ such that $\delta_1 + 1/\delta_2 < \delta$. We shall use the easily proved estimate

$$
B(k,q,r) \leqq 2d^*. \tag{6.3}
$$

We consider four parts $\sum_1, \sum_2, \sum_3, \sum_4$ of the sum we want to estimate.

$\sum_1 :\ d^* < q^{f-\varepsilon}$. We may assume $f > 1$, since $d^* < q^{f-\varepsilon}$ is otherwise impossible.

$$
\begin{aligned}
\textstyle\sum_1 &\leqq 2\sum_{q=1}^{h}\ \sum_{r\leqq q \text{ with } d^*<q^{f-}} \psi(q)P(q)^{-1}d^* \\
&\leqq 2\sum_{q=1}^{h}\psi(q)P(q)^{-1} \sum_{d|P(q);\ d<q^{f-}} d \sum_{r\leqq q;\ d|P(r)} 1.
\end{aligned}
$$

Using Lemma 6 and partial summation we obtain $\sum_1 \ll \Psi(h)$.

$\sum_2 :\ d^* < (q/r)^{1/\delta_1}$.

$$
\begin{aligned}
\sum_{r<qd^{*-\delta_1}} B(k,q,r) &\ll \sum_{d|P(q)}\ \sum_{r<qd^{-\delta_1};d|P(r)} d \leqq \sum_{d|P(q)} d \sum_{x\leqq P(qd^{-\delta_1});\,d|x} 1 \\
&\ll \sum_{d|P(q)} dP(q)d^{-1-f\delta_1}.
\end{aligned}
$$

$$
\textstyle\sum_2 \ll \sum_{q=1}^{h}\psi(q)\sum_{d|P(q)} d^{-f\delta_1} \ll \Psi(h)
$$

by Lemma 7 and partial summation.

$\sum_3: d^* < \Psi(q)^{\delta_2}$.

$$\sum_{r<q \text{ with } d^*<\Psi(q)^{\delta_2}} B(k,q,r) \ll \sum_{d|P(q);\, d<\Psi(q)^{\delta_2}} \sum_{r \leqq q;\, d|P(r)} d$$

$$\leqq \sum_{d|P(q);\, d<\Psi(q)^{\delta_2}} d \sum_{x \leqq P(q);\, d|x} 1$$

$$\leqq \sum_{d|P(q);\, d<\Psi(q)^{\delta_2}} dP(q)d^{-1} = P(q) \sum_{d|P(q);\, d<\Psi(q)^{\delta_2}} 1.$$

Putting $l = \Psi(h)^{\delta_2}$ we obtain

$$\sum_3 \ll \sum_{q=1}^{h} \psi(q) D_l(P(q)).$$

Lemma 8 together with partial summation gives $\sum_3 \ll \Psi(h)^{1+\delta}$.

$\sum_4: d^* \geqq q^{f-\varepsilon},\ d^* \geqq (q/r)^{1/\delta_1},\ d^* \geqq \Psi(q)^{\delta_2}$. Under these conditions, $P(q)\psi(r) \leqq P(q)\psi(qd^{*-\delta_1})$(3) $= P(q)q^{-1}d^{*\delta_1}qd^{*-\delta_1}\psi(qd^{*-\delta_1}) \leqq P(q)q^{-1}d^{*\delta_1}$ $\Psi(q) \leqq P(q)q^{-1}d^{*\delta_1+1/\delta_2} \leqq P(q)q^{-1}d^{*\delta}$; therefore $B(k,q,r) \leqq E_\delta(k,q,r)$. Obviously, $\sum_4$ is bounded by the right-hand sum of (6.2).

7. **Proof of Proposition 2 for nonlinear polynomials.** In the case of polynomials of degree $f > 1$ we may use Lemma 6, which ceases to be true if $f = 1$. On the other hand, much of the preceding discussion could be simplified for $f = 1$.

We assume now $f > 1$.

We define $(x, y; k)$ by

$$(x,y;k) = \begin{cases} \text{g.c.d.}(x,y) \text{ if this divisor is } \geqq xk^{-1}, \\ 0 \text{ otherwise.} \end{cases}$$

Lemma 12. *Assume that $\delta > 0$ is so small that $f - 1/2 < (f - 1/4)(1 - \delta)$. Further assume $q > q_0(P,\delta)$, $r \leqq q$, $d^* = \text{g.c.d.}(P(q),P(r)) \geqq q^{f-1/4}$. Then*

$$E_\delta(k,q,r) \leqq (P(q),P(r);k). \tag{7.1}$$

Proof. Put $c = |a_0| + \cdots + |a_f|$, where the a_i's are the coefficients of $P(q)$. Choose q_0 so large that the two inequalities

$$q_0^{1/4} > 2c, \quad 2cq_0^{f-1/2} < q_0^{(f-1/4)(1-\delta)}$$

hold, and let $q \geqq q_0$.

The numbers $a = a(q)$, $b = b(q)$ satisfy

$$\theta = b/a + R, \quad |R| < a^{-1}q^{-1/2}, \quad a \leqq q^{1/2}$$

and

$$2caq^{f-1} \leqq 2cq^{f-1/2} < q^{(f-1/4)(1-\delta)} \leqq d^{*1-\delta}. \tag{7.2}$$

(3) For $0 \leqq \alpha < 1$ define $\psi(n-\alpha) = \psi(n)$.

$E_\delta(k,q,r)$ is bounded by the number of pairs p,s, $0 \leqq p < P(q)$, $p \in S(k,q)$ satisfying

$$|P(q)s - P(r)p + (P(q) - P(r))(b/a + R)| < P(q)q^{-1}d^{*\delta}.$$

This equation together with (7.2) yields

$$\begin{aligned} |P(q)(s + b/a) - P(r)(p + b/a)| &< P(q)q^{-1}d^{*\delta} + |P(q) - P(r)||R| \\ &\leqq cq^{f-1}d^{*\delta} + cq^f a^{-1}q^{-1/2} \\ &< d^*/2a + q^{f-1/4}/2a \leqq d^*/a. \end{aligned} \tag{7.3}$$

The left-hand side of (7.3) is an integral multiple of d^*/a, hence it must be zero.

$$P(q)(as + b) = P(r)(ap + b). \tag{7.4}$$

It remains to estimate the number of solutions of (7.4) in pairs p,s, $0 \leqq p<P(q)$, $p \in S(k,q)$. Putting $P(q) = d^*P(q)^*$ we find that (7.4) implies

$$ap + b \equiv 0 (\operatorname{mod} P(q)^*).$$

Since a and b are relatively prime, this congruence has at most one solution in p modulo $P(q)^*$, hence at most d^* solutions in $0 \leqq p < P(q)$. On the other hand, the congruence gives g.c.d.$(ap + b, P(q)) \geqq P(q)^* = P(q)d^{*-1}$, while $p \in S(k,q)$ implies g.c.d.$(ap + b, P(q)) \leqq k$. Thus $E_\delta(k,q,r)$ is zero unless $d^* \geqq P(q)k^{-1}$.

Lemma 12 is proved.

Proof of Proposition 2 ($f > 1$). We may assume that $\delta > 0$ is so small that $f - 1/2 < (f - 1/4)(1 - \delta)$. Combining Lemma 10, Lemma 11 with $\varepsilon = 1/4$ and Lemma 12, we obtain

$$\sum_{q=1}^{h} \sum_{r=1}^{h} A(k,q,r) \ll \Psi(h)^{1+\delta} + \sum_{q=1}^{h} \sum_{r=1}^{q} \psi(q) P(q)^{-1}(P(q), P(r); k).$$

Using Lemma 8 and partial summation we find

$$\begin{aligned} \sum_{q=1}^{h} \sum_{r=1}^{q} P(q)^{-1}(P(q),P(r);k) &\leqq \sum_{q=1}^{h} P(q)^{-1} \sum_{d|P(q);\, d\geqq P(q)k^{-1}} d \sum_{r\leqq q;\, d|P(r)} 1 \\ &\leqq \sum_{q=1}^{h} P(q)^{-1} \sum_{d|P(q);\, d\geqq P(q)k^{-1}} dP(q)d^{-1} \\ &= \sum_{q=1}^{h} \sum_{d|P(q);\, d\leqq k} 1 \\ &= \sum_{q=1}^{h} D_k(P(q)) \ll hk^\delta \end{aligned}$$

and

$$\sum_{q=1}^{h} \sum_{r=1}^{q} \psi(q) P(q)^{-1}(P(q), P(r); k) \ll \Psi(h)k^\delta.$$

8. Proof of Proposition 2 for linear polynomials.

LEMMA 13. *Let $P(x)$ be a linear polynomial, let $0 < \delta < 1/4$, $q \geqq 1$ and $d^* \mid P(q)$. Then*

$$\sum_{r \leqq q;\, (P(q),\, P(r)) = d^*} E_\delta(k,q,r) \ll qd^{*-1/4} + d^* + q^{1/2} + \begin{cases} q \text{ if } d^* \geqq P(q)k^{-1}, \\ 0 \text{ otherwise.} \end{cases} \tag{8.1}$$

REMARK. The constant involved in the symbol $\ll$ depends on $P(x)$ only.

Proof. Put $c = |a_0| + |a_1|$ where $P(x) = a_0 x + a_1$. The relation $|P(q)s - P(r)p + (P(q) - P(r))\theta| < P(q)q^{-1}d^{*\delta} \leqq cd^{*1/4}$ in the definition of $E_\delta(k,q,r)$ implies $\|(P(q) - P(r))d^{*-1}\theta\| < cd^{*-3/4}$. Given an r such that the last inequality holds, there are at most $2c$ integers l with

$$|(P(q) - P(r))d^{*-1}\theta - l| < cd^{*-3/4}.$$

Given r and l, $P(q)s - P(r)p = ld^*$ has at most d^* solutions in p, $0 \leqq p < P(q)$. Putting

$$F(q,r) = \begin{cases} 1 \text{ if } \|(P(q) - P(r))d^{*-1}\theta\| < cd^{*-3/4}, \\ 0 \text{ otherwise,} \end{cases}$$

we thus find

$$E_\delta(k,q,r) \ll d^*F(q,r). \tag{8.2}$$

Assume now that r runs through those values between 1 and q where $d^* \mid P(r)$. Then $P(q) - P(r)$ runs through some or all of the numbers 0, d^*, $2d^*$, $\cdots$, $[a_0q/d^*]\,d^* = q^*d^*$. Thus if we put $G(q,d^*)$ for the number of integers x in $0 \leqq x \leqq q^*$ satisfying

$$\|x\theta\| < cd^{*-3/4}, \tag{8.3}$$

then

$$\sum_{r \leqq q;\, d^* \mid P(r)} F(q,r) \leqq G(q,d^*). \tag{8.4}$$

We now distinguish three cases: A, B and C.

A. $2ca(q) \geqq d^{*1/4}$. How often does (8.3) hold when x runs through an interval $m < x \leqq m + a(q)$? Putting $\theta = b/a + R$ and $x = m + y$, the inequality becomes $\|m\theta + yb/a + yR\| < cd^{*-3/4}$ and this implies $\|m\theta + yb/a\| < cd^{*-3/4} + a|R| < cd^{*-3/4} + q^{-1/2}$. The number of solutions of (8.3) for x in an interval of length a is therefore $\ll (d^{*-3/4} + q^{-1/2})a + 1 \ll d^{*-3/4}a + 1$. Hence

$$G(q,d^*) \ll (d^{*-3/4}a + 1)(q^*a^{-1} + 1) \ll q^*d^{*-1/4} + q^{1/2}d^{*-3/4} + 1,$$

and

$$d^*(G(q,d^*)) \ll qd^{*-1/4} + d^*.$$

(8.1) now follows from (8.2), (8.4) and the last inequality.

B. $2ca(q) < d^{*1/4}$, $2aq^*|R| \geqq 1$. Putting $\theta = b/a + R$ again, we rewrite (8.3) as $\| xb/a + xR \| < cd^{*-3/4}$. This implies that

$$|m/a + xR| < cd^{*-3/4} \tag{8.5}$$

for some integer m. For fixed m the number of solutions in x of (8.5) is at most $2cd^{*-3/4}|R|^{-1} + 1$. On the other hand, $x \leqq q^*$, whence $|m| \leqq cad^{*-3/4} + aq^*|R|$. Thus

$$\begin{aligned} G(q,d^*) &\ll (d^{*-3/4}|R|^{-1} + 1)(ad^{*-3/4} + aq^*|R| + 1) \\ &\leqq a^2q^*d^{*-3/2} + ad^{*-3/4} + aq^*d^{*-3/4} + q^*q^{-1/2} + 2aq^*d^{*-3/4} + 1 \\ &\ll a^2q^*d^{*-3/4} + q^*q^{-1/2} + 1 \end{aligned}$$

and

$$d^*G(q,d^*) \ll qd^{*-1/4} + q^{1/2} + d^*.$$

C. $2ca(q) < d^{*1/4}$, $2aq^*|R| < 1$. $E_\delta(k,q,r)$ is bounded by the number of solutions in integers p, s, $0 \leqq p < P(q)$, $p \in S(k,q)$, of

$$|P(q)(s + b/a) - P(r)\,(p + b/a) + (P(q) - P(r))R| < cd^{*1/4}.$$

Now for $r \leqq q$, $d^* | P(r)$, one has $|P(q) - P(r)| \leqq q^*d^*$, and we obtain the inequality

$$|P(q)(s + b/a) - P(r)(p + b/a)| \leqq cd^{*1/4} + q^*d^*|R| < d^*/2a + d^*/2a = d^*/a.$$

Just as in the proof of Lemma 12 we may conclude that (7.4) holds, and we obtain (7.1). The number of $r \leqq q$ with $d^* | P(r)$ is $\ll qd^{*-1}$, and therefore

$$\sum_{r \leqq q;\, d^* | P(r)} E_\delta(k,q,r) \ll \begin{cases} q & \text{if } d^* \geqq P(q)k^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Proposition 2 ($f = 1$). We may assume $0 < \delta < 1/4$. By applying Lemma 10 and Lemma 11 with $\varepsilon = 1$ we obtain

$$\sum_{q=1}^{h} \sum_{r=1}^{h} A(k,q,r) \ll \Psi(h)^{1+\delta} + \sum_{q=1}^{h} \sum_{r=1}^{q} \psi(q)P(q)^{-1}E_\delta(k,q,r).$$

By Lemma 13,

$$\begin{aligned} &\sum_{q=1}^{h} \sum_{r=1}^{h} P(q)^{-1}E_\delta(k,q,r) \\ &\ll \sum_{q=1}^{h} q^{-1} \sum_{d^*|P(q)} (qd^{*-1/4} + d^* + q^{1/2}) + \sum_{q=1}^{h} \sum_{d^*|P(q);\, d^* \geqq P(q)k^{-1}} 1 \\ &\ll \sum_{q=1}^{h} \sum_{d | P(q)} d^{-1/4} + \sum_{q=1}^{h} q^{-1/2}D(P(q)) + \sum_{q=1}^{h} D_k(P(q)) \\ &\ll h + h + hk^\delta \ll hk^\delta. \end{aligned}$$

Here we used (4.1) and Lemmas 7 and 8 to estimate the last three sums. Proposition 2 now follows by partial summation.

9. **The higher dimensional case.** Most of the arguments used for the case $n = 1$ carry over immediately to $n > 1$, but some of them have to be modified.

We may assume that I_{jq} is of the type $0 < \{\xi_j - \theta_j\} \leqq \psi_j(q)$. For each of the integers $j = 1, \cdots, n$ we can now define $a_j(q)$, $b_j(b)$, $S_j(k,q)$, $\beta_j(q,\alpha_j)$, $\gamma_j(q,\alpha_j), \cdots, \Gamma_j(k,q,r)$. For given q, r we write d_j^* for the greatest common divisor of $P_j(g)$ and $P_j(r)$, and we may now define $B_j(k,q,r), \cdots, E_{j\delta}(k,q,r)$, $F_j(q,r)$, $G_j(q,d_j^*)$. We put $\beta(q,\alpha_1,\cdots,\alpha_n) = \prod_j \beta_j(q,\alpha_j)$, $\gamma(q,\alpha_1,\cdots,\alpha_n) = \prod_j \gamma_j(q,\alpha_j), \cdots$, $\Gamma(k,q,r) = \prod_j \Gamma_j(k,q,r)$, and we define $A(k,q,r)$ as in paragraph 2.

PROPOSITION 1a. *Let* $\delta > 0$. *Then*

$$\sum_{q=1}^{h} (P_1(q)\cdots P_n(q) - \phi(k,P_1(q))\cdots \phi(k,P_n(q)))(P_1(q)\cdots P_n(q))^{-1} \ll hk^{\delta-1} + h^{\delta}k^{\delta}.$$

PROPOSITION 2a. *Let* $\delta > 0$. *Then* (3.2) *holds.*

The argument of paragraph 3 can be used to deduce the general theorem from these propositions.

Proposition 1a follows from Proposition 1 and

$$\begin{aligned} &P_1(q)\cdots P_n(q) - \phi(k,P_1(q))\cdots \phi(k,P_n(q)) \\ &\quad = \sum_{j=1}^{n} P_1 \cdots P_{j-1}(P_j - \phi(k,P_j))\phi(k,P_{j+1})\cdots \phi(k,P_n). \end{aligned}$$

(6.1) now becomes

$$\Gamma_j(k,q,r) \leqq \psi_j(q)\psi_j(r) + \psi_j(q)P_j(q)^{-1}B_j(k,q,r),$$

and therefore for $r \leqq q$

$$A(k,q,r) \leqq \sum_{m=1}^{n} \sum_{\Delta_m} H(k,q,r;m,\Delta_m),$$

where Δ_m runs through all divisions of the integers $1,\cdots,n$ into two classes $i_1,\cdots,i_m$ and $j_1,\cdots,j_{n-m}$, and where

$$H(k,q,r;m,\Delta_m) = \psi(q) \prod_{s=1}^{m} (P_{i_s}^{-1}(q)B_{i_s}(k,q,r)) \prod_{t=1}^{n-m} \psi_{j_t}(r).$$

For reasons of symmetry it will suffice to estimate $H(k,q,r;m,\Delta_m^0)$, where Δ_m^0 is the division with $i_1 = 1,\cdots, i_m = m$. We shall use

$$B_i(k,q,r) \leqq 2d_i^* \qquad (i = 1,\cdots,n). \tag{9.1}$$

LEMMA 14. *Let* $m > 1$. *Then*

$$\sum_{q=1}^{h} \sum_{r=1}^{q} H(k,q,r;m,\Delta_m^0) \ll \Psi(h).$$

Proof. We use the estimate

$$H(k,q,r;m,\Delta_m^0) \ll \psi(q)P_1(q)^{-1}P_2(q)^{-1}d_1^* d_2^*.$$

By Schwartz' inequality,

$$\begin{aligned} Y_h &\equiv \sum_{q=1}^{h}\sum_{r=1}^{q} P_1(q)^{-1}d_1^* P_2(q)^{-1}d_2^* \\ &\leqq \left(\sum_{q=1}^{h}\sum_{r=1}^{q} P_1(q)^{-2}d_1^{*2}\right)^{1/2}\left(\sum_{q=1}^{h}\sum_{r=1}^{q} P_2(q)^{-2}d_2^{*2}\right)^{1/2}. \end{aligned}$$

Now

$$\begin{aligned} \sum_{q=1}^{h}\sum_{r=1}^{q} P(q)^{-2}d^{*2} &\leqq \sum_{q=1}^{h} P(q)^{-2}\sum_{d\mid P(q)} d^2 \sum_{r\leqq q;\, d\mid P(r)} 1 \\ &\leqq \sum_{q=1}^{h} P(q)^{-2}\sum_{d\mid P(q)} d^2P(q)d^{-1} = \sum_{q=1}^{h}\sum_{d\mid P(q)} d^{-1} \ll h \end{aligned}$$

by Lemma 7. Hence $Y_h \ll h$, and Lemma 14 follows by partial summation.

Everything can be completed as in the case $n = 1$ once we have shown

LEMMA 11a. *Let* $\varepsilon = 1$ *if the degree* f_1 *of* $P_1(x)$ *equals* 1, $\varepsilon > 0$ *if* $f_1 > 1$. *Let* $\delta > 0$. *Then*

$$\begin{aligned} &\sum_{q=1}^{h}\sum_{r=1}^{q} H(k,q,r;1,\Delta_1^0) \\ &\qquad \ll \Psi(h)^{1+\delta} + \sum_{q=1}^{h} \sum_{r\leq q \text{ with } d_1^*\geq q^{f_1-\varepsilon}} \psi(q)P_1(q)^{-1}E_{1\delta}(k,q,r). \end{aligned}$$

Proof. Choose $\delta_1 > 0$, $\delta_2 > 0$ such that $\delta_1 + 1/\delta_2 < \delta$. We write $\chi_1(q) = \psi_2(q)\cdots\psi_n(q)$ and put

$$\Psi_1(h) = \sum_{q=1}^{h} \psi_1(q),$$

$$X_1(h) = \sum_{q=1}^{h} \chi_1(q).$$

Since both $\psi_1(q)$ and $\chi_1(q)$ are decreasing, one has

$$h\Psi(h) \geqq \Psi_1(h)\,\chi_1(h). \tag{9.2}$$

$H(k,q,r;1,\Delta_1^0)$ equals $\psi(q)P_1(q)^{-1}B_1(k,q,r)\chi_1(r)$.

We consider four parts of the sum we want to estimate. $\sum_1$ consists of terms with $d_1^* < q^{f_1-\varepsilon}$, $\sum_2$ of terms with $d_1^* < (q/r)^{1/\delta_1}$; $\sum_3$ of terms where $d_1^* < \Psi_1(q)^{\delta_2}$, and $\sum_4$ consists of the remaining terms, that is, terms where $d_1^* \geqq q^{f_1-\varepsilon}$,

$d_1^* \geqq (q/r)^{1/\delta_1}, d_1^* \geqq \Psi_1(q)^{\delta_2}$. For the parts $\sum_1$, $\sum_2$, $\sum_4$ we estimate $H(\cdots) \leqq \psi(q)P_1(q)^{-1}B_1(k,q,r)$ and proceed as in paragraph 6. The difficulty lies in estimating $\sum_3$.

Let $d|P_1(q)$ and denote the numbers r having $r \leqq q$ and $d|P_1(r)$ by $r_1 < r_2 < \cdots < r_j$. One has $j \leqq P_1(q)d^{-1}$ and $r_j \geqq c(jd)^{1/f_1} \geqq c'djP_1(q)^{-1}q$ for large q. Hence

$$\sum_{r \leqq q \text{ with } d_1^* < \Psi_1(q)}{}_{\delta_2} B_1(k,q,r)\chi_1(r) \ll \sum_{d|P_1(q);\, d<\Psi_1(q)}{}_{\delta_2} d \sum_{r \leqq q;\, d|P_1(r)} \chi_1(r)$$
$$\ll \sum_{d|P_1(q);\, d<\Psi_1(q)}{}_{\delta_2} d\,\chi_1(q)P_1(q)q^{-1}d^{-1}$$
$$= \chi_1(q)P_1(q)q^{-1}D_f(P_1(q))$$

with $f = f(q) = \Psi_1(q)^{\delta_2}$. Hence

$$\sum_3 \ll \sum_{q=1}^{h} \psi(q)q^{-1}\mathrm{X}_1(q)D_f(P_1(q)). \tag{9.3}$$

We estimate the last sum in three parts, which overlap somewhat.

$T_1: \Psi(q) \geqq q^{1/4\delta_2}$. Unless this part is empty, there is a largest $q \leqq h$ in T_1 say q_1. By (4.1),

$$T_1 \leqq \sum_{q=1}^{q_1} \psi(q)D(P_1(q)) \ll \sum_{q=1}^{q_1} \psi(q)q^{\delta/4\delta_2} \leqq \Psi(q_1)q_1^{\delta/4\delta_2}$$
$$\leqq \Psi(q_1)^{1+\delta} \leqq \Psi(h)^{1+\delta}.$$

$T_2: \chi_1(q) \leqq q^{1-1/4\delta_2}$. Again using (4.1) we obtain

$$T_2 \ll \sum_{q=1}^{h} \psi(q)q^{-1/4\delta_2}q^{1/4\delta_2} = \Psi(h).$$

$T_3: \Psi(q) < q^{1/4\delta_2}, X_1(q) > q^{1-1/4\delta_2}$. We denote the set of q's involved by σ, by h_1 the largest element of σ, and we write $g(q) = \psi(q)q^{-1}\chi_1(q)$. Observing (9.2) we find $\Psi_1(q) \leqq q^{1/2\delta_2}$ and $f(q) \leqq q^{1/2}$ for $q \in \sigma$.

$$T_3 = \sum_{q\in\sigma} g(q)D_f(P_1(q)) = \sum_{d \leqq f(h_1)} \sum_{q\in\sigma:\, d|P_1(q);\, f(q)\geqq d} g(q).$$

Let $x_1, \ldots, x_{z(d)}$ be the solutions of $P_1(x) \equiv 0 (\bmod d)$. Since $g(q)$ is decreasing, one has

$$\sum_{q\in\sigma;\, f(q)\geqq d;\, q\equiv x_i (\bmod d)} g(q) \leqq g(q_i) + d^{-1} \sum_{f(q)\geqq d} g(q),$$

where q_i is the smallest $q \equiv x_i \,(\bmod d)$ such that $q \in \sigma$ and $f(q) \geqq d$. Since $f(q) \leqq q^{1/2}$, one finds $q_i \geqq d^2$. Therefore

$$\sum_{q\in\sigma:\, f(q)\leqq d;\, d|P_1(q)} g(q) \leqq z(d)g(d^2) + z(d)d^{-1} \sum_{f(q)\geqq d} g(q).$$

Observing $z(d)g(d^2) \leqq dg(d^2) \leqq \sum_{q=(d-1)^2+1}^{d^2} g(q)$ we obtain

$$T_3 \leqq \sum_{d \leqq f(h_1)} \left(\sum_{q=(d-1)^2+1}^{d^2} g(q) + z(d)d^{-1} \sum_{f(q) \geqq d} g(q) \right)$$

$$\leqq \sum_{q=1}^{f^2(h_1)} \psi(q) + \sum_{q=1}^{h} g(q) \sum_{d \leqq f(q)} z(d)d^{-1}$$

$$\ll \Psi(h) + \sum_{q=1}^{h} g(q)\Psi_1(q)^{\delta}$$

by Lemma 5. (9.2) finally yields

$$T_3 \ll \Psi(h) + \sum_{q=1}^{h} \psi(q)(q^{-1}\chi_1(q))^{1-\delta}\Psi(q)^{\delta} \ll \Psi(h)^{1+\delta}.$$

Lemma 11a is proved.

10. **Linear forms.** We restrict ourselves to the case of one form only.

PROPOSITION 3. Let $P_1(q_1), \cdots, P_n(q_n)$ *be nonconstant polynomials,* $n+1$, *and let* $I_{q_1,\ldots,q_n}$ *be intervals of* C $(q_i = 1,2,\cdots;\ i = 1,\cdots,n)$. *We assume that the length of* $I_{q_1,\ldots,q_n}$ *is* $\psi_1(q_1)\psi_2(q_2)\cdots\psi_n(q_n)$, *where* $\psi_i(x)$ *are decreasing functions* $(i = 1,\cdots,n)$, *and we put*

$$\Psi_i(h) = \sum_{q_i=1}^{h} \psi_i(q_i).$$

We write $M(h_1,\cdots,h_n;\alpha_1,\cdots,\alpha_n)$ *for the number of solutions of* $\{\alpha_1 P_1(q_1) + \cdots + \alpha_n P_n(q_n)\} \in I_{q_1,\ldots,q_n}$, *where* $1 \leqq q_i \leqq h_i$ $(i = 1,\cdots,n)$. *Let* $\varepsilon > 0$. *Then for almost all* $\alpha_1,\cdots,\alpha_n$,

$$M(h_1,\cdots,h_n;\alpha_1,\cdots,\alpha_n) = \Psi_1(h_1)\cdots\Psi_n(h_n) + O(\Psi_1(h_1)\cdots\Psi_n(h_n))^{1/2+\varepsilon}.$$

This estimate holds simultaneously for $h_1,\cdots,h_n$.

Proof. We restrict ourselves to a few hints. The reader might compare paragraph 6 of [12]. We assume $n > 1$.

We put $\beta(q_1,\cdots,q_n,\xi)$ equal to 1 if $\{\xi\} \in I_{q_1,\ldots,q_n}$ and $\xi \in U$, $\beta(\cdots) = 0$ otherwise. $\Gamma(q_1,\cdots,q_n;r_1,\cdots,r_n)$ stands for the integral

$$\int_0^1 \cdots \int_0^1 \left(\sum_p \left(\beta q_1,\cdots,q_n, \sum \alpha_i P_i(q_i) - p \right) \right) \left(\sum_s \beta \left(r_1,\cdots,r_n, \sum \alpha_i P_i(r_i) - s \right) \right) d\alpha_1 \cdots d\alpha_n,$$

and $A(q_1,\cdots,q_n;\, r_1,\cdots,r_n)$ for

$$\Gamma(q_1,\cdots,r_n) - \psi_1(q_1)\cdots\psi_n(q_n)\psi_1(r_1)\cdots\psi_n(r_n).$$

PROPOSITION 2b. $\sum_{q_1=1}^{h_1} \cdots \sum_{r_n=1}^{h} A(q_1,\cdots,r_n) \ll \Psi_1(h_1)\cdots\Psi_n(h_n)$.

To deduce Proposition 3 from Proposition 2b we put

$$M(h_1,\cdots,h_n;\ k_1,\cdots,k_n;\alpha_1,\cdots,\alpha_n)$$

for the number of $q_1,\cdots,q_n$, $h_i < q_i \leqq k_i$ $(i=1,\cdots,n)$ such that $\{\sum\alpha_i P_i(q_i)\} \in I_{q_1,\dots,q_n}$ and we put $\Psi_i(h,k) = \sum_{h<q\leqq k}\psi_i(q)$. We choose integers $m_u^i = m_u^i(r_1,\cdots,r_n)$ such that $[2^{r_1+\cdots+r_n-r_i}\Psi_i(m_u^i)] = u$ The following two lemmas are now used.

Lemma 2b. *Let $\delta > 0$. Then one has for $T = T_{r_1+\dots+r_n}$*

$$\sum_{(u_1v_1]\in J}\cdots\sum_{(u_nv_n]\in J}\int_0^1 (M(m_{u_1}^1,\cdots,m_{u_n}^n;m_{v_n}^1,\cdots,m_{v_n}^n;\alpha_1,\cdots,\alpha_n)$$
$$-\Psi_1(m_{u_1}^1,m_{v_1}^1)\cdots\Psi_n(m_{u_n}^n,m_{v_n}^n))^2 d\alpha_1\cdots d\alpha_n$$
$$\ll 2^{(r_1+\dots+r_n)(1+\delta)}.$$

Lemma 3b. *Let $\delta > 0$. There are subsets $\sigma_{r_1,\dots,r_n}(r_i = 1,2,\cdots;\ i = 1,\cdots,n)$ of $U\times\cdots\times U$ with measures*

$$\mu_{r_1,\dots,r_n} \ll r_1^{-2}\cdots r_n^{-2}$$

such that

$$M(m_{w_1}^1,\cdots,m_{w_n}^n;\alpha_1,\cdots,\alpha_n) = \Psi_1(m_{w_1}^1)\cdots\Psi_n(m_{w_n}^n) + O(r_1^2\cdots r_n^2 2^{(r_1+\dots+r_n)(1/2+\delta)})$$

for every $w_1,\cdots,w_n$ with $w_i \leqq 2^{r_1+\cdots+r}$ $(i=1,\cdots,n)$ and $(\alpha_1,\cdots,\alpha_n)$ in $U\times\cdots\times U$ but not in $\sigma_{r_1,\dots,r_{..}}$.

To prove Proposition 2b we require

Lemma 10b. A. *If the matrix*

$$\begin{pmatrix} P_1(q_1),\cdots,P_n(q_n) \\ P_1(r_1),\ \cdots,P_n(r_n) \end{pmatrix}$$

has rank 2, then

$$A(q_1,\cdots,r_n) = 0.$$

B. *If the matrix has rank 1, then*

$$A(q_1,\cdots,r_n) \leqq \psi_1(q_1)\cdots\psi_n(q_n)\,P_1(q_1)^{-1}B_1(q_1,\cdots,r_n),$$

where $B_1(q_1,\cdots,r_n)$ is the number of solutions of $|P_1(q_1)(s+\theta')-P_1(r_1)(p+\theta)| < d_1^*$ in integers $p,s,0 \leqq p < P_1(q_1)$, where θ, θ' are the left endpoints of $I_{q_1,\dots,q_n}$, $I_{r_1,\dots,r_n}$, respectively, and where $d_1^* =$ g.c.d.$(P_1(q_1),P_1(r_1))$.

We leave the proof of A to the reader. As for B, we make the substitution $\alpha_2 = \xi_2,\cdots,\alpha_n = \xi_n$, $\sum\alpha_i P_i(q_i) = \xi_1 P_1(q_1)$, hence $\sum\alpha_i P_i(r_i) = \xi_1 P_1(r_1)$. When $\xi_2,\cdots,\xi_n$ is fixed, ξ_1 ranges in an interval of length 1, and Γ equals

$$\int_0^1 \Big(\sum_p \beta(q_1,\cdots,q_n,\xi_1P_1(q_1)-p)\Big)\Big(\sum_s \beta(r_1,\cdots,r_n,\xi_1P_1(r_1)-s)\Big)d\xi_1.$$

This one-dimensional integral can be estimated by the method of paragraph 6.

The proof of Proposition 2b now proceeds as follows. We may restrict ourselves to terms $r_1 \leqq q_1$. For fixed $q_1,\cdots,q_n$, let $\Delta = \text{g.c.d.}(P_1(q_1),\cdots,P_n(q_n))$ and $P_i(q_i) = P_i(q_i)^*\Delta$ $(i=1,\cdots,n)$. In view of Lemma 10b we may restrict ourselves to $r_1 \leqq q_1$ where $P_1(r_1)$ is of the type $lP_1(q_1)^*$, whence $d_1^* = P_1(q_1)^*(\Delta,l)$. Since $B(q_1,\cdots,r_n) \leqq 2d_1^*$, one has

$$\begin{aligned}\sum_{q_1=1}^{h_1}\cdots\sum_{r_n=1}^{h_n} A(q_1,\cdots,r_n) &\leqq 4\sum_{q_1=1}^{h_1}\cdots\sum_{q_n=1}^{h_n}\psi_1(q)_1\cdots\psi_n(q_n)\sum_{l=1}^{\Delta} P_1(q_1)^{-1}(P_1(q_1)^*(\Delta,l))\\ &\leqq 4\sum_{q_1=1}^{h_1}\cdots\sum_{q_n=1}^{h_n}\psi_1(q_1)\cdots\psi_n(q_n)D(\Delta)\\ &\leqq 4\Big(\sum_{q_1=1}^{h_1}\sum_{q_2=1}^{h_2}\psi_1(q_1)\psi_2(q_2)D(P_1(q_1),P_2(q_2))\Big)\\ &\qquad\qquad\Psi_3(h_3)\cdots\Psi_n(h_n).\end{aligned}$$

Using Lemma 9 and partial summation both for the sum over q_1 and over q_2 we obtain

$$\sum_{q_1=1}^{h_1}\sum_{q_2=1}^{h_2}\psi_1(q_1)\psi_2(q_2)D(P_1(q_1),P_2(q_2)) \ll \Psi_1(h_1)\Psi_2(h_2).$$

11. **Theorem 2.** To prove the lower bound in (1.5) we shall need

PROPOSITION 4. *Let $a(1) < a(2) < \cdots$ be a sequence of positive integers and put $M_I(h;\alpha)$ for the number of $q \leqq h$ such that $\{\alpha a(q)\} \in I$. Then for $\varepsilon > 0$ and almost all α the inequality*

$$|M_I(h;\alpha) - hl(I)| < h^{1/2}\log^{5/2+\varepsilon}h$$

holds for all intervals I and all $h > h_1$, where h_1 depends only on α and ε (but not on I).

Proof. This proposition is a special case of Theorem 1 of [3] and of Theorem 1 of [6].

Proof of Theorem 2. We use the abbreviation

$$\|\Sigma\| = \Big\|\sum_{i=1}^n \alpha_i a_i(q_i) + \theta\Big\|.$$

Put $\delta = \varepsilon/(n+1)$. Using an idea of Littlewood [4, Appendix A], we consider the integral

$$J(q_1,\cdots,q_n) = \int_0^1\cdots\int_0^1 (\|\Sigma\|\,|\log\|\Sigma\||^{1+\delta})^{-1}d\alpha_1\cdots d\alpha_n.$$

This integral has a finite value independent of $q_1, \cdots, q_n$. Hence the sum

$$\sum_{q_1=1}^{\infty} \cdots \sum_{q_n=1}^{\infty} (q_1 \log^{1+\delta} q_1 \cdots q_n \log^{1+\delta} q_n)^{-1} J(q_1, \cdots, q_n)$$

is convergent and

$$(11.1) \qquad \sum_{q_1=1}^{\infty} \cdots \sum_{q_n=1}^{\infty} (q_1 \log^{1+\delta} q_1 \cdots q_n \log^{1+\delta} q_n \| \Sigma \| |\log \| \Sigma \| |^{1+\delta})^{-1}$$

is convergent for almost all $\alpha_1, \cdots, \alpha_n$.

It is easy to see that the inequality

$$(11.2) \qquad \| \Sigma \| \leqq (q_1 \cdots q_n)^{-2}$$

has only a finite number of solutions in integers $q_1, \ldots, q_n$ for almost every $\alpha_1, \cdots, \alpha_n$.

We are going to show that the upper estimate for $\Sigma(h; \alpha_1, \cdots, \alpha_n)$ in (1.5) is true for every $\alpha_1, \cdots, \alpha_n$ such that (11.1) is convergent and (11.2) has only finitely many solutions. There is a constant $c > 0$ such that $\| \Sigma \| \geqq c^{-1}(q_1 \cdots q_n)^{-2}$, whence

$$|\log \| \Sigma \| | \leqq 2 \log (q_1 \cdots q_n) + \log c.$$

We obtain

$$\Sigma(h; \alpha_1, \cdots, \alpha_n) \leqq (\max_{q_i \leqq h} (\log^{1+\delta} q_1 \cdots \log^{1+\delta} q_n |\log \| \Sigma \| |^{1+\delta}))$$

$$\cdot \sum_{q_1=1}^{\infty} \cdots \sum_{q=1}^{\infty} (q_1 \log^{1+\delta} q_1 \cdots q_n \log^{1+\delta} q_n \| \Sigma \| |\log \| \Sigma \| |^{1+\delta})^{-1}$$
$$\ll (\log h)^{(1+\delta)n} (\log h^n)^{1+\delta} \ll (\log h)^{n+1+\varepsilon}.$$

We now turn to the proof of the lower bound in (1.5). We are going to apply Proposition 4 to the sequence $a(q) = a_1(q)$. For almost all reals α and $h \geqq h_1(\alpha)$ one has $|M_I(h; \alpha) - hl(I)| < h^{3/4}$. Let α_1 have this property. Denote the number of $q \leqq h$ such that $\| \alpha_1 a_1(q) + \eta \| \leqq \gamma$ by $M_{\gamma,\eta}(h; \alpha_1)$. Then

$$|M_{\gamma,\eta}(h; \alpha_1) - 2\gamma h| < h^{3/4} \qquad (h \geqq h_1(\alpha_1), 0 \leqq \gamma \leqq 1/2, \eta \text{ arbitrary}).$$

Let $k_0 = k_0(h)$ be the largest integer with $2^{k_0+1} \leqq h^{1/4}$. Then $k_0 \geqq 0$ for $h \geqq h_2(\alpha_1) = \max(h_1(\alpha_1), 2^8)$. The number $N_{k,\eta}(h; \alpha_1)$ of $q \leqq h$ such that

$$(11.3) \qquad 2^{-k-1} < \| \alpha_1 a_1(q) + \eta \| \leqq 2^{-k}$$

satisfies $N_{k,\eta}(h; \alpha_1) \geqq 2^{-k} h - 2h^{3/4} \geqq 2^{-k-1} h$ for every k in $0 \leqq k \leqq k_0$.

By considering the parts of the sum where (11.3) is satisfied for $k = 0, \cdots, k_0$ we obtain

$$\sum_{q=1}^{h} \| \alpha_1 a_1(q) + \eta \|^{-1} \geqq \sum_{k=0}^{k_0} 2^k 2^{-k-1} h > \tfrac{1}{2} h k_0(h) \geqq c_1(\alpha_1) h \log h.$$

Partial summation yields

$$\sum_{q=1}^{h} \| \alpha_1 a_1(q) + \eta \|^{-1} q^{-1} \geqq c_2(\alpha_1) \log^2 h.$$

This inequality holds for arbitrary η. By writing $\eta = \alpha_2 a_2(q_2) + \cdots + \alpha_n a_n(q_n) + \theta$ and taking the sum over $q_2, \cdots, q_n$ one finds

$$\sum(h; \alpha_1, \cdots, \alpha_n) \geqq c_3(\alpha_1) \log^{n+1} h.$$

REMARK. Our method could be used to show the following: The left inequality of (1.5) is true for arbitrary $\alpha_1, \cdots, \alpha_{n-1}$ and $\alpha_n \in \sigma(\alpha_1, \cdots, \alpha_{n-1})$, where $\sigma(\cdots)$ is a set containing almost all numbers. The other inequality of (1.5) holds for n-tuples such that $\alpha_n \in \tau$ where τ is independent of $\alpha_1, \cdots, \alpha_{n-1}$ and contains almost all numbers.

12. **Theorem 3.** We define a function $\pi(k_1, \cdots, k_n)$ as follows. $\pi(0, \cdots, 0) = 0$, and if $k_{i_1}, \cdots, k_{i_m}$ are those k_i's which are different from zero, then $\pi(k_1, \cdots, k_n) = |k_{i_1} \cdots k_{i_m}|^{-1}$. In our applications $k_1, \cdots, k_n$ will always be integers. Write $\exp \xi$ for $e^{2\mu i \xi}$.

GENERALIZED THEOREM OF ERDÖS AND TURAN. *There are absolute constants c_n, $n = 1, 2, \cdots$ with the following properties.*

Let $n \geqq 1$, $h \geqq 1$, and let vectors $(\alpha_{1q}, \cdots, \alpha_{nq})$ be given $(q = 1, \cdots, h)$. Put

$$\omega(k_1, \cdots, k_n) = \left| \sum_{q=1}^{h} \exp \left(\sum_{i=1}^{n} \alpha_{iq} k_i \right) \right|.$$

Let $I_1, \cdots, I_n$ be intervals of C of lengths $l(I_j) = \psi_j$ and put $\psi = \prod \psi_j$. Write N for the number of q, $1 \leqq q \leqq h$, such that simultaneously $\{\alpha_{jq}\} \in I_j (j = 1, \cdots, n)$. Let m be a positive integer. Then

$$|N - \psi l| \leqq c_n \left(h\bar{m}^{1} + \sum_{k_1, \ldots, k_n : |k_j| \leqq m} \pi(k_1, \cdots, k_n) \omega(k_1, \cdots, k_n) \right).$$

This theorem is a generalization to n dimensions of a result of Erdös and Turan [7, Theorem 3]. We shall not give a proof, since the argument in [7] can easily be extended to our situation.

Proof of Theorem 3. Put $\alpha_{jq} = \alpha_j q \; (j = 1, \cdots, n; \; q = 1, 2, \cdots)$.

$$\omega_h(k_1, \cdots, k_n) = \left| \sum_{q=1}^{h} \exp(\sum k_i \alpha_i q) \right| \ll \| k_1 \alpha_1 + \cdots + k_n \alpha_n \|^{-1}.$$

Theorem 3 is an immediate consequence of the generalized Erdös-Turan Theorem with $m = h$ and the fact that

$$\sum_{k_1, \ldots, k_n : |k_i| \leqq m} \pi(k_1, \cdots, k_n) \| k_1 \alpha_1 + \cdots + k_n \alpha_n \|^{-1} \ll (\log m)^{n+1+\varepsilon}$$

for almost every $\alpha_1 \cdots, \alpha_n$. This fact follows from Theorem 2.

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